Simplifying the Calculation of Limits

For many functions like $f(x) = 2$, $g(x) = x$, and $h(x) = 3x - 2$, we guess that the limit as x approaches a is the function evaluated at a, i.e., $\lim_{x\to a} f(x) = f(a)$. This turns out to be true many times. It would be convenient to not have to prove this for every problem.
To develop an easier way to work some limit problems we note that if $\lim_{x\to a} f(x) = f(a)$ then f is continuous at a, because this is the definition of continuity. Thus we want to know if certain, common functions are continuous. Below we prove that a number of types of functions are continuous.

Theorem 1 *Constant functions are continuous.*

Let $f(x) = K$ where K is some real number. We consider $\lim_{x\to a} f(x)$ for an arbitrary real number a. If f is continuous, then the limit is K.

$$\begin{aligned} |f(x) - K| &< \epsilon. \\ |K - K| &< \epsilon. \\ |0| &< \epsilon. \\ 0 &< \epsilon. \end{aligned}$$

This statement is always true, regardless of what value we pick for δ. Therefore $\lim_{x\to a} K = K$ for all a. This is equivalent to stating that a constant function is continuous everywhere.

Theorem 2 *The function $g(x) = x$ is continuous.*

We consider $\lim_{x\to a} g(x)$ for an arbitrary real number a. If g is continuous, then the limit is a.

$$\begin{aligned} |g(x) - a| &< \epsilon. \\ |x - a| &< \epsilon. \\ \delta &= \epsilon. \end{aligned}$$

Thus $\lim_{x\to a} x = a$ for all a. This is equivalent to stating that the function $g(x) = x$ is continuous everywhere.

Theorem 3 *All polynomials are continuous.*

A polynomial is a function $p(x) = a_k x^k + \ldots + a_2 x^2 + a_1 x + a_0$. We must show that $\lim_{x\to a} p(x) = p(a)$ for all real values a.

$$\begin{aligned} \lim_{x\to a} p(x) &= \lim_{x\to a} a_k x^k + \ldots + a_2 x^2 + a_1 x + a_0 \\ &= \lim_{x\to a} a_k x^k + \ldots + \lim_{x\to a} a^2 x^2 + \lim_{x\to a} a_1 x + \lim_{x\to a} a_0 && (1) \\ &= a_k \lim_{x\to a} x^k + \ldots + a^2 \lim_{x\to a} x^2 + a_1 \lim_{x\to a} x + \lim_{x\to a} a_0 && (2) \\ &= a_k \left(\lim_{x\to a} x\right)^k + \ldots + a^2 \left(\lim_{x\to a} x\right)^2 + a_1 \left(\lim_{x\to a} x\right) + \lim_{x\to a} a_0 && (3) \\ &= a_k a^k + \ldots a_2 a^2 + a_1 a + a_0 && (4) \\ &= p(a). \end{aligned}$$

Step 1 uses the summation tool for limits. Step 2 uses the limit tool for constant multipliers. Step 3 uses the multiplication tool for limits. Finally step 4 uses that the functions $f(x) = K$ and $g(x) = x$ are continuous everywhere.
Thus we have shown $\lim_{x\to a} p(x) = p(a)$ for all polynomials and all real values a. This is equivalent to stating that all polynomials are continuous everywhere.