

If $h(x) = g(f(x))$, $\lim_{x \rightarrow a} f(x) = L$, and $\lim_{x \rightarrow L} g(x) = M$, then $\lim_{x \rightarrow a} g(f(x)) = M$. Below is an explanative example.

Let $f(x) = 2x$, $g(x) = x^2$, $h(x) = g(f(x)) = (2x)^2$.

1. We show first that $\lim_{x \rightarrow 3} f(x) = 6$.

$$\begin{aligned}
|2x - 6| &< \epsilon_f. \\
-\epsilon_f &< 2x - 6 < \epsilon_f. \\
-\epsilon_f + 6 &< 2x < \epsilon_f + 6. \\
-\frac{\epsilon_f}{2} + 3 &< x < \frac{\epsilon_f}{2} + 3. \\
-\frac{\epsilon_f}{2} + 3 - 3 &< x - 3 < \frac{\epsilon_f}{2} + 3 - 3. \\
-\frac{\epsilon_f}{2} &< x - 3 < \frac{\epsilon_f}{2}. \\
|x - 3| &< \frac{\epsilon_f}{2}. \\
\delta_f &= \frac{\epsilon_f}{2}.
\end{aligned}$$

2. Next we show that $\lim_{x \rightarrow 6} g(x) = 36$.

$$\begin{aligned}
|x^2 - 36| &< \epsilon_g. \\
-\epsilon_g &< x^2 - 36 < \epsilon_g. \\
-\epsilon_g + 36 &< x^2 < \epsilon_g + 36. \\
\sqrt{-\epsilon_g + 36} &< x < \sqrt{\epsilon_g + 36}. \\
\sqrt{-\epsilon_g + 36} - 6 &< x - 6 < \sqrt{\epsilon_g + 36} - 6. \\
|x - 6| &< \sqrt{\epsilon_g + 36} - 6. \\
\delta_g &= \sqrt{\epsilon_g + 36} - 6.
\end{aligned}$$

3. Finally we want to show that $\lim_{x \rightarrow 3} g(f(x)) = 36$.

For an arbitrary ϵ we need to find a δ such that $|(2x)^2 - 36| < \epsilon$ when $|x - 3| < \delta$. From part 2 we know that $|(2x)^2 - 36| < \epsilon$ when $|2x - 6| < \sqrt{\epsilon + 36} - 6$. From part 1 we know that $|2x - 6| < \sqrt{\epsilon + 36} - 6$ when $|x - 3| < \frac{\sqrt{\epsilon + 36} - 6}{2}$. Thus for an arbitrary ϵ we choose $\delta = \frac{\sqrt{\epsilon + 36} - 6}{2}$.