Calculus II

Final Exam Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Please begin each section of questions on a new sheet of paper.
3. Please do not write answers side by side.
4. Please do not staple your test papers together.
5. Limited credit will be given for incomplete or incorrect justification.
6. All numeric answers must be exact: numeric approximations are not acceptable.

Questions

1. Integration (5 each)

   (a) $\int \sin^7 \alpha \cos^3 \alpha \, d\alpha$

$$\begin{aligned} \int \sin^7 \alpha \cos^3 \alpha \, d\alpha &= \quad \begin{array}{rcl} u &=& sin\alpha. \\ du &=& \cos \alpha \, d\alpha. \end{array} \\ \int \sin^7 \alpha (1 - \sin^2 \alpha) \cos \alpha \, d\alpha &= \\ \int u^7 (1 - u^2) \, du &= \\ \int u^7 - u^9 \, du &= \frac{1}{8} u^8 - \frac{1}{10} u^{10} + C \\ &= \frac{1}{8} \sin^8 \alpha - \frac{1}{10} \sin^{10} \alpha + C. \end{aligned}$$

   (b) $\int (3x^2 + 5x - 1) \sin x \, dx$

$$\begin{aligned} \int (3x^2 + 5x - 1) \sin x \, dx &= \quad \begin{array}{rclrcl} u &=& 3x^2 + 5x - 1. & dv &=& \sin x \, dx. \\ du &=& 6x + 5 \, dx. & v &=& -\cos x. \end{array} \\ -(3x^2 + 5x - 1) \cos x - \int -(6x + 5) \cos x \, dx &= \\ -(3x^2 + 5x - 1) \cos x + \int (6x + 5) \cos x \, dx &= \quad \begin{array}{rclrcl} u &=& 6x + 5. & dv &=& \cos x \, dx. \\ du &=& 6 \, dx & v &=& \sin x. \end{array} \\ -(3x^2 + 5x - 1) \cos x + (6x + 5) \sin x - \int 6 \sin x \, dx &= \\ -(3x^2 + 5x - 1) \cos x + (6x + 5) \sin x + 6 \cos x + C. \end{aligned}$$

(c) $\displaystyle\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$

$$\begin{aligned}
\int \frac{x^2}{(1-x^2)^{3/2}}\,dx &= \quad \begin{aligned} x &= \sin\theta. \\ dx &= \cos\theta\,d\theta. \end{aligned} \\
\int \frac{\sin^2\theta}{(1-\sin^2\theta)^{3/2}}\cos\theta\,d\theta &= \\
\int \frac{\sin^2\theta}{\cos^3\theta}\cos\theta\,d\theta &= \\
\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta &= \\
\int \tan^2\theta\,d\theta &= \\
\int \sec^2\theta - 1\,d\theta &= \tan\theta - \theta + C \\
&= \frac{x}{\sqrt{1-x^2}} - \arcsin x + C.
\end{aligned}$$

(d) $\displaystyle\int \arctan y\,dy$

$$\begin{aligned}
\int \arctan y\,dy &= \quad \begin{aligned} u &= \arctan y. & dv &= dy. \\ du &= \tfrac{dy}{1+y^2}. & v &= y. \end{aligned} \\
y\arctan y - \int \frac{y}{1+y^2}\,dy &= \quad \begin{aligned} u &= 1+y^2. \\ du &= 2y\,dy. \end{aligned} \\
y\arctan y - \frac{1}{2}\int \frac{2y}{1+y^2}\,dy &= \\
y\arctan y - \frac{1}{2}\int \frac{du}{u} &= y\arctan y - \frac{1}{2}\ln|1+y^2| + C.
\end{aligned}$$

(e) $\displaystyle\int \frac{6z-11}{z^2+z-12}\,dz$

$$\begin{aligned}
\frac{6z-11}{z^2+z-12} &= \\
\frac{6z-11}{(z+4)(z-3)} &= \frac{A}{z+4} + \frac{B}{z-3}. \\
6z-11 &= A(z-3) + B(z+4). \\
z &= 3. \\
7 &= 7B. \\
B &= 1. \\
z &= -4. \\
-35 &= -7A. \\
A &= 5.
\end{aligned}$$

$$\begin{aligned}
\int \frac{6z-11}{z^2+z-12}\,dz &= \\
\int \frac{5}{z+4} + \frac{1}{z-3}\,dz &= 5\ln|z+4| + \ln|z-3| + C.
\end{aligned}$$

(f) $\displaystyle\int \frac{x^3+x-7}{x^2+1}\,dx$

$$\begin{array}{r|rrrr} & x & & & \\ \hline x^2+0+1 & x^3 & +0 & +x & -7 \\ & x^3 & +0 & +x & \\ \cline{2-4} & & & & -7 \end{array}$$

$$\begin{aligned} \int \frac{x^3+x-7}{x^2+1}\,dx &= \\ \int x - \frac{7}{x^2+1}\,dx &= \frac{1}{2}x^2 - 7\arctan x + C. \end{aligned}$$

(g) $\displaystyle\int_0^\infty re^{-r^2}\,dr$

$$\begin{aligned} \int_0^\infty re^{-r^2}\,dr &= \\ \lim_{a\to\infty}\int_0^a re^{-r^2}\,dr &= \qquad \begin{aligned} u &= -r^2. \\ du &= -2r\,dr. \end{aligned} \\ \lim_{a\to\infty} -\frac{1}{2}\int_0^a -2re^{-r^2}\,dr &= \\ \lim_{a\to\infty} -\frac{1}{2}\int_0^{-a^2} e^u\,du &= \\ \lim_{a\to\infty} -\frac{1}{2}e^u\Big|_0^{-a^2} &= \\ \lim_{a\to\infty} -\frac{1}{2}(e^{-a^2}-1) &= \frac{1}{2}. \end{aligned}$$

2. Series (5 each)
   Determine if each of the following converge or diverge. State the test(s) used.

   (a) $\sum_{n=1}^{\infty} \frac{5^n + n}{5^{3n}}$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5^n + n}{5^{3n}} &= \\
\sum_{n=1}^{\infty} \frac{5^n}{5^{3n}} + \sum_{n=1}^{\infty} \frac{n}{5^{3n}} &= \\
\sum_{n=1}^{\infty} \frac{1}{5^{2n}} + \sum_{n=1}^{\infty} \frac{n}{5^{3n}} &= \\
\sum_{n=1}^{\infty} \frac{1}{(5^2)^n} + \sum_{n=1}^{\infty} \frac{n}{5^{3n}}. & \\
\lim_{n\to\infty} \left(\frac{n}{5^{3n}}\right)^{1/n} &= \\
\lim_{n\to\infty} \frac{n^{1/n}}{5^3} &= \frac{1}{125} \\
&< 1.
\end{aligned}$$

   Thus the first term is a convergent geometric series and the second is convergent by the root test. Thus the sum is convergent.

   (b) $\sum_{n=1}^{\infty} \left(\frac{2n^2}{3n^2+1}\right)$

   Using the nth term divergence test

$$\begin{aligned}
\lim_{n\to\infty} \frac{2n^2}{3n^2+1} &\overset{\infty/\infty}{=} \quad \text{L'Hôpital's Rule} \\
\lim_{n\to\infty} \frac{4n}{6n} &= \\
\lim_{n\to\infty} \frac{2}{3} &= \tfrac{2}{3}.
\end{aligned}$$

   Thus this series diverges.

   (c) $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

   Using the ratio test

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} &= \\
\lim_{n\to\infty} \frac{(n+1)}{2} &= \infty.
\end{aligned}$$

   This series diverges.

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

Using the limit comparison test

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} \div \frac{1}{\sqrt{n}} &= \\
\lim_{n\to\infty} \frac{n}{n+1} &\overset{\infty/\infty}{=} \quad \text{L'Hôpital's Rule} \\
\lim_{n\to\infty} 1 &= 1.
\end{aligned}$$

This means this series diverges like the p-series with $p = 1/2$.

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+1}$

Using the alternating series test

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} &\overset{\infty/\infty}{=} \quad \text{L'Hôpital's Rule} \\
\lim_{n\to\infty} \frac{1}{2\sqrt{n}} &= 0. \\
a_n &= \frac{\sqrt{n}}{n+1}. \\
a_n' &= \frac{(n+1)n^{-1/2}/2 - \sqrt{n}}{(n+1)^2} \\
&= \frac{1-n}{2\sqrt{n}(n+1)^2}.
\end{aligned}$$

Thus the series converges.

(f) $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n^{2n}}$

Using absolute convergence with the root test

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{5^n}{n^{2n}}\right)^{1/n} &= \\
\lim_{n\to\infty} \frac{5}{n^2} &= 0 \\
&< 1.
\end{aligned}$$

Thus this series converges.

3. Power Series

   (a) Find the interval of convergence for each of the following. (6 each)

      i. $\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$
      Using the ratio test

$$\begin{aligned} \lim_{n\to\infty} \frac{x^{3(n+1)}}{(3[n+1])!} \cdot \frac{(3n)!}{x^{3n}} &= \\ \lim_{n\to\infty} \frac{x^{3n+3}}{(3n+3)!} \cdot \frac{(3n)!}{x^{3n}} &= \\ \lim_{n\to\infty} \frac{x^3}{(3n+3)(3n+2)(3n+1)} &= 0. \end{aligned}$$

      This converges

$$(-\infty, \infty).$$

      ii. $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$
      Using the ratio test

$$\begin{aligned} \lim_{n\to\infty} \frac{x^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{x^n} &= \\ \lim_{n\to\infty} x\frac{2n+1}{2n+3} &\overset{\infty/\infty}{=} \text{L'Hôpital's Rule} \\ \lim_{n\to\infty} x &= x \\ &< 1. \\ \sum_{n=0}^{\infty} \frac{1^n}{2n+1} &= \sum_{n=0}^{\infty} \frac{1}{2n+1}. \\ \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} & \\ \lim_{n\to\infty} \frac{1}{2n+1} &= 0. \\ \frac{1}{2(n+1)+1} &< \frac{1}{2n+1}. \end{aligned}$$

      Thus the interval of convergence is

$$[-1, 1).$$

iii. $\sum_{n=0}^{\infty} \frac{(7x-3)^n}{3}$

Using the root test

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{(7x-3)^n}{3}\right)^{1/n} &= \\
\lim_{n\to\infty} \frac{7x-3}{3^{1/n}} &= 7x-3 \\
&< 1.
\end{aligned}$$

$$\begin{array}{rcll}
-1 & \leq & 7x-3 & \leq 1. \\
2 & \leq & 7x & \leq 4. \\
\frac{2}{7} & \leq & x & \leq \frac{4}{7}.
\end{array}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(7(4/7)-3)^n}{3} &= \sum_{n=0}^{\infty} \frac{1}{3}. \\
\sum_{n=0}^{\infty} \frac{(7(2/7)-3)^n}{3} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{3}.
\end{aligned}$$

Both ends diverge (nth term divergence). The interval of convergence is

$$\left(\frac{2}{7}, \frac{4}{7}\right).$$

(b) Write a power series representation for each of the following. Be sure to write in correct form. (4 each)

i. $\frac{1+x}{x} e^x$

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\frac{1}{x} e^x &= \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots \\
\frac{1+x}{x} e^x &= \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots \\
& \quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
&= \frac{1}{x} + 2 + \frac{3x}{2!} + \frac{4x^2}{3!} + \dots
\end{aligned}$$

ii. $\sinh x$

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2}. \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \\
\sinh x &= \frac{1}{2}\left(2x + 2\frac{x^3}{3!} + \dots\right) \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots
\end{aligned}$$

iii. $x\cos(x^2)$

$$
\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\cos(x^2) &= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots \\
x\cos(x^2) &= x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \dots
\end{aligned}
$$

4. Modeling

   (a) $$\begin{aligned} x &= t + \sin(2t). \\ y &= t - 2\sin t. \\ t &\in [0, 4\pi]. \end{aligned}$$

      i. (4) Find all times when the curve changes from left to right or vice versa.

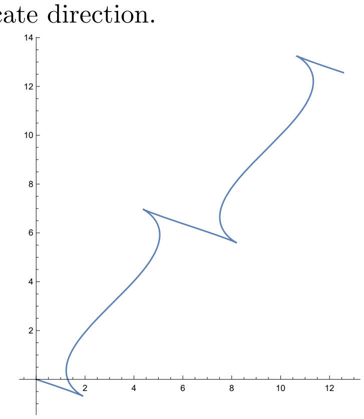
$$\begin{aligned} x' &= 1 + 2\cos(2t). \\ 1 + 2\cos(2t) &= 0. \\ 2\cos(2t) &= -1. \\ \cos(2t) &= -1/2. \\ 2t &= \ldots, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \ldots \\ t &= \ldots, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots \end{aligned}$$

      ii. (4) Find all times when the curve changes from up to down or vice versa.

$$\begin{aligned} y' &= 1 - 2\cos t. \\ 1 - 2\cos t &= 0. \\ \cos t &= 1/2. \\ t &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots \end{aligned}$$

      iii. (2) Where are the non-differentiable points? (Give t values.)
      x' and y' are both zero at $\ldots, \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$

      iv. (2) Graph the curve. Indicate direction.



      v. (2) Calculate the slope of the tangent at $t = \pi/2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 - 2\cos t}{1 + 2\cos(2t)}. \\ \left.\frac{dy}{dx}\right|_{t=\pi/2} &= \frac{1 - 2\cos(\pi/2)}{1 + 2\cos(2[\pi/2])} \\ &= -1. \end{aligned}$$

(b) $r = \sin(\theta) + \sin(2\theta)/2$.

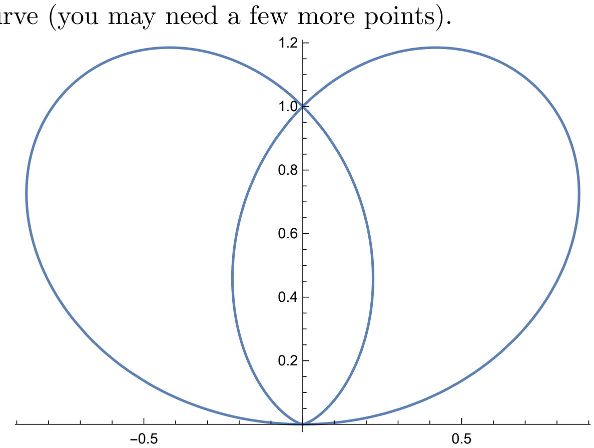
i. (4) Determine when the curve is at the origin.

$$\begin{aligned}
\sin(\theta) + \sin(2\theta)/2 &= 0. \\
\sin(\theta) + \sin\theta\cos\theta &= 0. \\
\sin(\theta)(1 + \cos\theta) &= 0. \\
\sin\theta &= 0. \\
\theta &= \ldots, 0, \pi, 2\pi, \ldots \\
\cos\theta &= -1. \\
\theta &= \ldots, \pi, 3\pi, 5\pi, \ldots
\end{aligned}$$

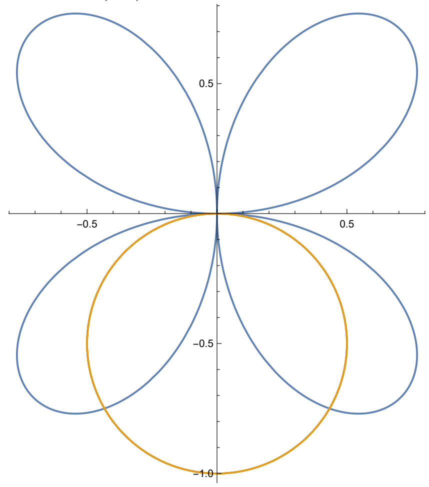
ii. (4) Determine when the curve changes from going out to coming in or vice versa.

$$\begin{aligned}
r' &= \cos\theta + \cos(2\theta) \\
&= \cos\theta + 2\cos^2\theta - 1. \\
2\cos^2\theta + \cos\theta - 1 &= 0. \\
(2\cos\theta - 1)(\cos\theta + 1) &= 0. \\
\cos\theta &= 1/2. \\
\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots \\
\cos\theta &= -1. \\
\theta &= \ldots, \pi, 3\pi, 5\pi, \ldots
\end{aligned}$$

iii. (2) Graph the curve (you may need a few more points).



(c) (6) Find the area inside both $r = \sin(2\theta)$ and $r = -\sin\theta$. Do not evaluate integrals.



$$\begin{aligned} \sin(2\theta) &= 0. \\ 2\theta &= \ldots, 0, \pi, 2\pi, 3\pi, \ldots \\ \theta &= \ldots, 0, \pi/2, \pi, 3\pi/2, \ldots \end{aligned}$$

$$\begin{aligned} \sin(2\theta) &= -\sin\theta. \\ 2\sin\theta\cos\theta &= -\sin\theta. \\ \sin\theta &= 0. \\ \theta &= \ldots, 0, \pi, 2\pi, \ldots \\ 2\cos\theta &= -1. \\ \cos\theta &= -1/2. \\ \theta &= \ldots, \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots \\ \text{Also} & \\ -\sin(2(\pi+\theta)) &= -\sin\theta. \\ -\sin(2\pi+2\theta) &= -\sin\theta. \\ -[\sin(2\pi)\cos(2\theta) + \sin(2\theta)\cos(2\pi)] &= -\sin\theta. \\ -\sin(2\theta) &= -\sin\theta. \\ -2\sin\theta\cos\theta &= -\sin\theta. \\ 2\cos\theta &= 1. \\ \cos\theta &= 1/2. \\ \theta &= \ldots, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots \end{aligned}$$

$$2\left[\int_0^{\pi/3} \frac{1}{2}(-\sin\theta)^2\, d\theta + \int_{4\pi/3}^{3\pi/2} \frac{1}{2}\sin^2(2\theta)\, d\theta\right]$$

5. Regional Data (4 each)
   Setup but do not integrate to find the following. The region is enclosed by $y = x^2$ and $y = x$.

   (a) The volume produced by rotating the region around the x-axis

   $$\int_0^1 \pi(x^2 - (x^2)^2)\, dx$$

   (b) The volume produced by rotating the region around the y-axis

   $$\int_0^1 2\pi x(x - x^2)\, dx$$

   (c) The volume produced by rotating the region around the $y = 2$

   $$\int_0^1 \pi([2 - x^2]^2 - [2 - x]^2)\, dx$$

   (d) The perimeter of the region

   $$\sqrt{2} + \int_0^1 \sqrt{1 + (2x)^2}\, dx$$