Math 231 Introduction to Discrete Mathematics
Exam 2 Key

Instructions

1. Do NOT write your answers on these sheets. Nothing written on the test papers will be graded.
2. Please begin each section of questions on a new sheet of paper.
3. Please do not write answers side by side.
4. Please do not staple your test papers together.
5. Limited credit will be given for incomplete or incorrect justification.

Questions

1. (4) For each relation in Figure 1 determine if they are reflexive, symmetric, anti-symmetric, and transitive. If a relation has a property, indicate this. If it does not, show why.

   (a) $X = \{a, b, c, d, e\}$. $R_1 = \{ (a, a), (a, b), (a, e), (b, b), (b, e), (c, c), (c, d), (d, d), (e, e) \}$.
   - Reflexive: Yes, all five are present
   - Symmetric: No (a, b) is present but not (b, a).
   - Anti-Symmetric: Yes, no non-trivial, symmetric pairs.
   - Transitive: Yes, all sets are present.

   (b) $X = \{a, b, c, d, e\}$. $R_2 = \{ (a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d) \}$.
   - Reflexive: No, (e, e) is not present.
   - Symmetric: Yes, all pairs are present.
   - Anti-Symmetric: No, (a, b) and (b, a) are present but $a \neq b$.
   - Transitive: Yes, all sets are present.

   (c) $X = \{a, b, c, d, e\}$. $R_1 = \{ (a, a), (b, b), (c, c), (d, d), (e, e) \}$.
   - Reflexive: Yes, all five pairs are present.
   - Symmetric: Yes, albeit trivially.
   - Anti-Symmetric: Yes, vacuously.
   - Transitive: Yes, vacuously.

2. Equivalence Relations

(a) (5) Prove that the following is an equivalence relation. Two complex numbers, $a + bi$ and $c + di$, are related if $\|a + bi\| = \|c + di\|$. Note $\|a + bi\| = \sqrt{a^2 + b^2}$.

The relation is reflexive.

Proof: Note $\|a + bi\| = \|a + bi\|$ so $a + bi$ is related to itself. □

The relation is symmetric.

Proof: If $a + bi$ is related to $c + di$ then $\|a + bi\| = \|c + di\|$. But also $\|c + di\| = \|a + bi\|$ by symmetry of $=$. Thus $c + di$ is related to $a + bi$. □

The relation is transitive.

Proof: If $a + bi$ is related to $c + di$ and $c + di$ is related to $e + fi$, then

$$\begin{aligned} \|a + bi\| &= \|c + di\|. \\ \|c + di\| &= \|e + fi\|. \\ \|a + bi\| &= \|e + fi\|. \end{aligned}$$

The last is by transitivity of $=$. Thus $a + bi$ is related to $e + fi$. □

(b) Find the equivalence classes of $i, 1 + i, 3 + 4i$. Show at least four elements.

- $[i] = \{\ldots, 1, i, -1, -i, \ldots\}$
- $[1 + i] = \{\ldots, 1 + i, 1 - i, -1 + i, -1 - i, 2, 2i, \ldots\}$
- $[3 + 4i] = \{\ldots, 3 + 4i, 3 - 4i, -3 + 4i, 3 - 4i, 5, -5, 5i, -5i, \ldots\}$

3. Partially Ordered Sets

(a) (5) Prove that the following is a poset. Two complex numbers, $a + bi$ and $c + di$, are related if $a \le c$ and $b \le d$.

The relation is reflexive.
Proof: Note $a \le a$ and $b \le b$, so $a + bi$ is related to itself. □

The relation is anti-symmetric.
Proof: If $a + bi$ and $c + di$ are related both ways then

$$\begin{aligned} a &\le c. \\ c &\le a. \\ a &= c. \\ b &\le d. \\ d &\le d. \\ b &= d. \end{aligned}$$

Thus $a + bi = c + di$ and the relation is anti-symmetric. □

The relation is transitive.
Proof: If $a + bi$ is related to $c + di$, and $c + di$ is related to $e + fi$ then

$$\begin{aligned} a &\le c. \\ c &\le e. \\ a &\le e. \\ b &\le d. \\ d &\le f. \\ b &\le f. \end{aligned}$$

Thus $a + bi$ is related to $e + fi$. □

(b) (2) List two comparable elements and two incomparable elements.

$1 + 2i$ and $2 + 3i$ are comparable.
$1 + 2i$ and $2 + i$ are incomparable.

(c) (2) State the lub and glb of $1 + 3i$ and $2 - 3i$.

$$\text{lub}(1 + 3i, 2 - 3i) = 2 + 3i.$$
$$\text{glb}(1 + 3i, 2 - 3i) = 1 - 3i.$$

(d) (3) Is the poset in Figure 2 a lattice? Explain.

No. a and b have no lub. Both c and d are upper bounds but neither is a bound of the other.

4. Combinatorics (3 each)

   (a) How many ways can a music device play three of twenty songs in shuffle mode (random selection with no repeating of songs)?

   $$\frac{20!}{17!}$$

   (b) How many ways can three of twenty people be selected to receive free tickets to an event?

   $$\frac{20!}{17!3!}$$

   (c) How many words can be constructed by re-arranging the letters of "audacious"?
   Because there are two a's and two u's

   $$\frac{9!}{2!2!}$$

   (d) How many sets of size three can be produced from a set of size ten?

   $$\frac{10!}{7!3!}$$

1. $X = \{a, b, c, d, e\}$. $R_1 = \{$ (a, a), (a, b), (a, e), (b, b), (b, e), (c, c), (c, d), (d, d), (e, e) $\}$.
2. $X = \{a, b, c, d, e\}$. $R_2 = \{$ (a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d) $\}$.
3. $X = \{a, b, c, d, e\}$. $R_1 = \{$ (a, a), (b, b), (c, c), (d, d), (e, e) $\}$.

Figure 1: Relations

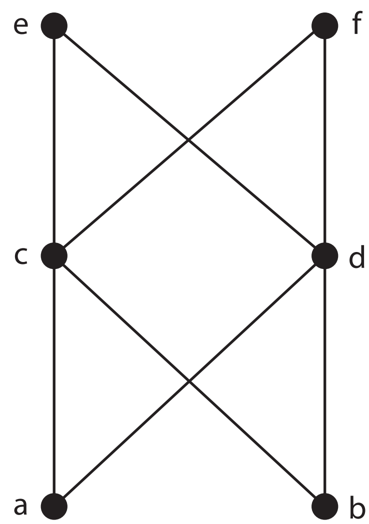

Figure 2: Hasse Diagram