The Laguerre-Pólya Class

Non-linear operators and the Riemann Hypothesis

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Theorem

The Riemann Hypothesis holds if and only if ξ belongs to the Laguerre-Pólya class.

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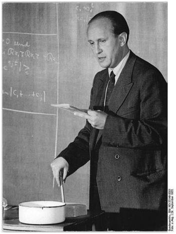
...and only the functions in the Laguerre-Pólya class enjoy this property.

The Turán Inequalities (1948)

A necessary condition.



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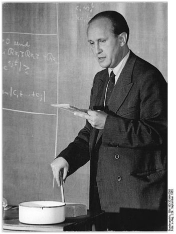

Theorem

If $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ *is a function in the Laguerre-Pólya class, then* $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$ *for* $k = 1, 2, 3, \ldots$ *.*

The Riemann ξ Function

$$\xi(x/2) = 8\int_0^\infty \Phi(t)\cos xt\, dt,$$

$$\Phi(t) = \sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}\right) e^{-n^2\pi e^{4t}}.$$

Theorem (Csordas, Varga, Norfolk (1986))

The coefficients of the Riemann ξ function satisfy the Turán inequalities.

Example

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\begin{aligned}
p(x) &= (x+1)(x+1)(x+1)(x+1)(x+1) \\
&= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5
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- Replace a_k with $3a_k^2 - 4a_{k-1}a_{k+1} + a_{k-2}a_{k+2}$.

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- Replace a_k with $3a_k^2 - 4a_{k-1}a_{k+1} + a_{k-2}a_{k+2}$.
- $p(x)$ becomes $3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5$.

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- $p(x)$ becomes $3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5$.
- ...and the zeros remain real and negative.

Example (continued...)

$$\begin{aligned} p(x) &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^5 + a_5 x^5 \ . \end{aligned}$$

The zeros of $p(x)$ remain real and negative if a_k is replaced with:

- $a_k^2 - a_{k-1} a_{k+1}$,
- $3a_k^2 - 4a_{k-1} a_{k+1} + a_{k-2} a_{k+2}$,
  - $p(x)$ becomes $3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5$

Example (continued...)

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The zeros of $p(x)$ remain real and negative if a_k is replaced with:

- $a_k^2 - a_{k-1}a_{k+1}$,
  - $p(x)$ becomes $1+15x+50x^2+50x^3+15x^4+x^5$
- $3a_k^2 - 4a_{k-1}a_{k+1} + a_{k-2}a_{k+2}$,
  - $p(x)$ becomes $3+35x+105x^2+105x^3+35x^4+3x^5$

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  - $p(x)$ becomes $3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5$
- $10a_k^2 - 15a_{k-1}a_{k+1} + 6a_{k-2}a_{k+2} - a_{k-3}a_{k+3}$,

Example (continued...)

$$p(x) = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$
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  - $p(x)$ becomes $3 + 35x + 105x^2 + 105x^3 + 35x^4 + 3x^5$
- $10a_k^2 - 15a_{k-1} a_{k+1} + 6a_{k-2} a_{k+2} - a_{k-3} a_{k+3}$,
  - $p(x)$ becomes $10 + 100x + 280x^2 + 280x^3 + 100x^4 + 10x^5$

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- $10a_k^2 - 15a_{k-1}a_{k+1} + 6a_{k-2}a_{k+2} - a_{k-3}a_{k+3}$,
  - $p(x)$ becomes $10 + 100x + 280x^2 + 280x^3 + 100x^4 + 10x^5$
- and infinitely many others....

The Main Result

Theorem (Grabarek (2010))

Let $\varphi(x) = \sum_{k=0}^{\omega} a_k x^k, 0 \leq \omega \leq \infty$, *be a function in the Laguerre-Pólya class. If the zeros of* $\varphi(x)$ *are real and negative, then the zeros remain real and negative after replacing* a_k *with*

$$\binom{2p-1}{p} a_k^2 + \sum_{j=1}^{p} (-1)^j \binom{2p}{p-j} a_{k-j} a_{k+j} \ \ (p = 1, 2, 3, \ldots) .$$