A New Class of Non-Linear Stability Preserving Operators

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Theorem (Pólya-Schur, 1914)

$\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}$-$\mathcal{P}$ *if and only if*

$$\varphi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}} \;,$$

where $m \in \mathbb{N}$, $\alpha, \beta, c, x_k \in \mathbb{R}$, $\alpha \geq 0$, *and* $\sum_{j=0}^{\infty} \frac{1}{x_k^2} < \infty$.

A restriction to real negative zeros.

$\mathscr{L}$-$\mathscr{P}^{+}$ consists of precisely those $\varphi(x) \in \mathscr{L}$-$\mathscr{P}$ whose Taylor coefficients are non-negative. Functions belonging to this subclass of $\mathscr{L}$-$\mathscr{P}$ are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials all of whose zeros are real and negative.

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$\mathscr{L}\text{-}\mathscr{P}^+$ consists of precisely those $\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}$ whose Taylor coefficients are non-negative. Functions belonging to this subclass of $\mathscr{L}\text{-}\mathscr{P}$ are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials all of whose zeros are real and negative.

Theorem (Pólya-Schur, 1914)

$\varphi(x) \in \mathscr{L}\text{-}\mathscr{P}^+$ if and only if

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) ,$$

where $m \in \mathbb{N}$, $\sigma, c, x_k \in \mathbb{R}$, $\sigma, c \geq 0$, $x_k > 0$, and $\sum_{j=0}^{\infty} \frac{1}{x_k} < \infty$.

Properties of the Laguerre-Pólya class

If $\psi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P}$, then the *Laguerre inequalities* hold: $(\psi^{(p)}(x))^2 - \psi^{(p-1)}(x)\psi^{(p+1)}(x) \geq 0$ for each $p = 1, 2, \ldots$, and all $x \in \mathbb{R}$.

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Theorem (Craven, Csordas, Patrick, Varga)

Let $\psi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ *be a real entire function and define*

$$L_p(\psi(x)) := \sum_{j=0}^{2p} \frac{(-1)^{p+j}}{(2p)!} \binom{2p}{j} \psi^{(j)}(x)\psi^{(2p-j)}(x) \ ,$$

where $x \in \mathbb{R}$ *and* $p = 0, 1, 2, \ldots$ *. Then* $\psi(x) \in \mathscr{L}\text{-}\mathscr{P}$ *if and only if for all* $x \in \mathbb{R}$ *and all* $p = 0, 1, 2, \ldots$

$$L_p(\psi(x)) \geq 0 \ .$$

Hadamard Composition

Let $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}$, $\psi(x) = \sum_{k=0}^{\infty} b_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$.

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Example (A non-linear operator acting on the coefficients)

Let $\varphi(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathscr{L}\text{-}\mathscr{P}^+$. The Hadamard composition is a linear operation, but we may regard $\varphi * \varphi(x) = \sum_{k=0}^{\infty} a_k^2 x^k \in \mathscr{L}\text{-}\mathscr{P}^+$ as the image of φ under the non-linear operator $a_k \mapsto a_k^2$, acting on the coefficients.

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- Are there other such non-linear operators?
- Can we characterize them?

Iterated Turán inequalities

Problem (Craven, Csordas, 1989)

Classify the functions

$$\psi(x) = \sum_{k=1}^{\omega} \frac{\gamma_k}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P} ,$$

where $\gamma_k \geq 0$ *and* $0 \leq \omega \leq \infty$, *for which the functions*

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k^2 - \gamma_{k-1}\gamma_{k+1}}{k!} x^k \in \mathscr{L}\text{-}\mathscr{P} .$$

A conjecture of Stanley, McNamara-Sagan, Fisk

Theorem (Brändén, 2009)

If the zeros of the real polynomial $\psi(x) = \sum_{k=0}^{n} a_k z^k$ are all real and negative, then the zeros of the polynomial

$$\sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1}) z^k \,, \text{ where } a_{-1} := 0 \text{ and } a_{n+1} := 0 \,,$$

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  - Shift indices and set $\gamma_k = 0$, whenever the index does not make sense. In general, for positive integers p

$$\frac{(2p)!}{2} L_p\left(\psi^{(k)}(x)\right)\Big|_{x=0} = \binom{2p-1}{p}\gamma_{k+p}^2 + \sum_{j=1}^{p}(-1)^j \binom{2p}{p-j}\gamma_{k+p-j}\gamma_{k+p+j} .$$

An extension

Theorem (Grabarek, 2010)

Let $\psi(z) = \sum_{k=0}^{n} a_k z^k = \prod_{k=1}^{n}(1+\rho_k z)$, *where* $\rho_k > 0$ *for* $1 \leq k \leq n$, *be a real polynomial with all real negative zeros. Let p be a positive integer and let* L_k^p *be the non-linear operator*

$$a_k \mapsto \binom{2p-1}{p} a_k^2 + \sum_{j=1}^{p} (-1)^j \binom{2p}{p-j} a_{k-j} a_{k+j} \; .$$

Then, the zeros of the polynomial

$$L_k^p [\psi(z)] = \sum_{k=0}^{n} L_k^p z^k$$

are all real and negative.

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  - Use the complex version of the *Laguerre inequalities*

$$|f'(z)|^2 \geq \Re\{f(z)\overline{f''(z)}\} .$$

Remarks and Further Direction

- In the proof of the Theorem, the reality of zeros of $L_k^p[\psi(z)]$, for each positive integer p, depends on the reality of zeros of the polynomial

$$Q_n^p(z) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{S(p,k)}{2} \binom{n}{2k} z^k \text{ , where } S(p,k) = \frac{\binom{2p}{p}\binom{2k}{k}}{\binom{p+k}{p}} \text{ .}$$

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  - Use *multiplier sequences* and *complex zero decreasing sequences* to obtain further properties.

Coefficient Bounds

Corollary (Grabarek)

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$$z \cdot \frac{\mu_2}{2} n(n-1)\ {}_2F_1\left(\frac{3}{2} - \frac{n}{2}, 1 - \frac{n}{2}; 3; 4z\right) + \mu_0\ {}_2F_1\left(\frac{1}{2} - \frac{n}{2}, -\frac{n}{2}; 2; 4z\right).$$

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  - The zeros of

$$ {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; p+1; 4z\right) $$

    are real and negative for any positive integer p and a fixed positive integer n (cf. K. Driver and K. Jordaan).

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    $$ {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; p+1; 4z\right) $$
    are real and negative for any positive integer p and a fixed positive integer n (cf. K. Driver and K. Jordaan).
  - Use contiguous function relations and/or the differential operator $\theta = z\frac{d}{dz}$.

- Classify the coefficients $\mu_0, \mu_2, \ldots, \mu_{2p-2}$ for which the non-linear operator L_k^p preserves $\mathscr{L}$-$\mathscr{P}^+$, or the reality of zeros.
- Use hypergeometric polynomials, rather than complex zero decreasing sequences.
    - The zeros of
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    are real and negative for any positive integer p and a fixed positive integer n (cf. K. Driver and K. Jordaan).
    - Use contiguous function relations and/or the differential operator $\theta = z\frac{d}{dz}$.
    - Iterates of
    $$ \mu \to {}_2F_1\left(-\frac{n}{2} + \mu, \frac{1-n}{2} + \mu; p+1; 4z\right) $$
    (notation of D. Karp) arise in the polynomials associated with L_k^p for $p \geq 2$.

Finally, regarding the non-linear operator $S_r : a_k \mapsto a_k^2 - a_{k-r}a_{k+r}$, we know from R. Yoshida's work that S_6 does not preserve $\mathscr{L}\text{-}\mathscr{P}^+$. However, we know that L_k^p does preserve $\mathscr{L}\text{-}\mathscr{P}^+$ for all positive integers p. We make the observation that

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and ask what is happening here?

Stay tuned for the 19th ICFIDCAA !