Poincaré's Homoclinic Horror

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The research story

A simplified mathematical model, in the form of a discrete dynamical system, was developed to study the spatially-localized, synchronous spiking observed in a large network of neural tissue. The model involved populations of excitatory and inhibitory neurons, interacting via synapses, and driven by a localized current input. Given that activity patterns of large populations of coupled neurons are notoriously difficult to study analytically, the motivation of this project was to develop an analytically-tractable approach to describe the spatial structure of the synchronous oscillations as the system parameters are varied. The simplified mathematical model is an implicit discrete map with two variables describing, on each cycle of a gamma frequency oscillation (30 – 90 Hz), the spatial extent of a synchronous spiking activity in the excitatory and inhibitory neuronal populations, respectively.

The image

The image visualizes a solution to the discrete map as ordered pairs in the phase plane, generated from 1 million iterations of the map. A chaotic solution occurs in the presence of a double homoclinic tangle, an extraordinarily intricate structure, arising in a 1:2 resonance bifurcation (codimension 2 bifurcation) as parameters are varied. Henri Poincaré first described the homoclinic tangle in 1899, after a glimpse in 1890, in the context of the famous three-body problem. Without the aid of computers to visualize such complicated solutions, he understood – from the mathematics alone – the implications such a structure would have on solutions and their characterization, referring to it in "horror." He, in fact, had discovered a primary mechanism that gives rise to chaos.

Reference

[1] Folias SE, Ermentrout GB, Spatially localized synchronous oscillations in synaptically coupled neuronal networks: conductance-based models and discrete maps. SIAM J. Applied Dynamical Systems 9: 1019–1060, 2010.

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